

Cohomology of Commuting Varieties of Connected Compact Reductive Lie Groups

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August 8, 2016

1 Abstract

We calculate the rational cohomology of the commuting variety $X_{G,n}$ consisting of n -tuples of commuting elements of a compact reductive group G . This is done by studying a map from a related variety $Y_{G,n}$, which has easily calculated cohomology. The proof studies the fibers of the map and uses the Vietoris-Begle theorem to prove that the induced map on rational cohomology is an isomorphism.

2 Introduction

Let G be a connected compact reductive group. Then the n th commuting variety $X_{G,n}$ is the variety consisting of all n -tuples $(g_1, g_2, \dots, g_n) \in G \times G$ that pairwise commute (i.e. $g_i g_j = g_j g_i$ for all i, j). Let T be a maximal torus, and let $N_G(T)$ denote the normalizer of T in G . Then let $Y_{G,n} := (G \times T^n)/N_G(T)$, where $N_G(T)$ acts by right-multiplication on G and by conjugation on T . Let $f : Y_{G,n} \rightarrow X_{G,n}$, $f(g, t'_1, t'_2, \dots, t'_n) = (gt'_1 g^{-1}, gt'_2 g^{-1}, \dots, gt'_n g^{-1})$; note that f is a G -equivariant map where G acts on $Y_{G,n}$ by acting on the factor of G by left-multiplication, and on $X_{G,n}$ by simultaneous conjugation.

Theorem 1 (Main Theorem). *The map f induces an isomorphism on rational cohomology, i.e. $H^*(X_{G,n}, \mathbb{Q}) \xrightarrow[f^*]{} H^*(Y_{G,n}, \mathbb{Q})$*

This theorem is a generalization of two already-known theorems.

Theorem 2. *Let G be a connected compact reductive group, T a maximal torus, and $N_G(T)$ the normalizer of T in G . Then $G/N_G(T)$ has trivial rational cohomology.*

Theorem 3. *Let G be a connected compact reductive group, T a maximal torus, and $N_G(T)$ the normalizer of T . Then $f : (G \times T)/N_G(T) \rightarrow G$ induces an isomorphism on rational cohomology.*

Proofs of both of these can be found in [1] (in the proof of Proposition 1). These can be seen as the $n = 0$ and $n = 1$ case of the main theorem, respectively.

This theorem allows relatively simple computation of the cohomology of the n th commuting variety. $X_{G,n}$ can be rewritten as $(G/T \times T^n)/W$, where W is the Weyl group of G . The action of W on G/T is free, so the action of W on $G/T \times T^n$ is free. Therefore, the cohomology of $X_{G,n}$ can be given by the W -invariants in the cohomology of $G/T \times T^n$. As the cohomology of G/T is known (if the grading is ignored, it is the regular representation of W), and the cohomology of T is isomorphic to the exterior algebra on the reflection representation of W , the cohomology is easy to calculate.

3 Proof of Main Theorem

We rely on a theorem from algebraic topology to reduce the question to studying the fibers of f .

Theorem 4 (Vietoris-Begle Theorem). *Let $f : Y \rightarrow X$ be a surjective map of compact metric spaces such that for all $x \in X$, $f^{-1}(x)$ is cohomologically trivial (with respect to some cohomology theory). Then f induces an isomorphism on cohomology (for the same cohomology theory).*

As all elements of a compact group are diagonalizable, any commuting n -tuple is contained in some maximal torus. All maximal tori are conjugate, so f is surjective. By Theorem 4 using rational cohomology, we only need to prove the following lemma:

Lemma 5. *For any commuting n -tuple (g_1, g_2, \dots, g_n) , the set $\{(g, t'_1, t'_2, \dots, t'_n) \in G \times T^n \mid \forall i, g t'_i g^{-1} = g_i\} / N_G(T)$ has trivial rational cohomology.*

The rest of the paper will prove this lemma by rewriting this set until it is in a form known to have trivial rational cohomology.

We can assume without loss of generality that the commuting n -tuple (g_1, g_2, \dots, g_n) is contained in our chosen maximal torus T . Change notation so that our commuting n -tuple is (t_1, t_2, \dots, t_n) . Let $X = \{(g, t'_1, t'_2, \dots, t'_n) \mid \forall i, g t'_i g^{-1} = t_i\}$; then $f^{-1}(t_1, t_2, \dots, t_n) = X / N_G(T)$.

Lemma 6. *Let G be a (not necessarily connected) reductive group. The G -orbit of an n -tuple of elements $(t_1, t_2, \dots, t_n) \in T^n$ meets T^n in exactly the $N_G(T)$ -orbit of (t_1, t_2, \dots, t_n) . In other words, if $g(t_1, t_2, \dots, t_n)g^{-1} = (t'_1, t'_2, \dots, t'_n) \in T^n$, then there is some $g' \in N_G(T)$ with $g'(t_1, t_2, \dots, t_n)g'^{-1} = (t'_1, t'_2, \dots, t'_n)$.*

Proof. We first reduce to the case that G is connected. Let $g \in G$ such that $g(t_1, t_2, \dots, t_n)g^{-1} = (t'_1, t'_2, \dots, t'_n) \in T^n$. Let $T' = gTg^{-1}$. As all maximal tori are conjugate by an element of the connected component of the identity G_0 , there is some $g_0 \in G_0$ such that $g_0 T g_0^{-1} = T'$. Then let $g_1 = g_0 g^{-1}$; an easy calculation shows that $g_1 \in N_G(T)$. As such, $g_1 t'_i g_1^{-1} \in T^n$, so let $t''_i = g_1 t'_i g_1^{-1}$. We then have that $g_0(t_1, t_2, \dots, t_n)g_0^{-1} = (t''_1, t''_2, \dots, t''_n)$. If

the theorem is true for connected G , then there is some $g_2 \in N_{G_0}(T)$ with $g_2(t_1, t_2, \dots, t_n)g_2^{-1} = (t_1'', t_2'', \dots, t_n'')$. Let $g' = g_1^{-1}g_2$; then $g'(t_1, t_2, \dots, t_n)g'^{-1} = g_1^{-1}(t_1'', t_2'', \dots, t_n'')g_1 = (t_1', t_2', \dots, t_n')$. We therefore only need to prove this in the case that G is connected.

The $n = 1$ case is a consequence of Chevalley's theorem. We prove this in the $n = 2$ case; the general case is similar, and works by induction. The general strategy is to reduce to the case that $t'_i = t_i$ for $i > 1$ by the inductive assumption, and then to reduce to the $n = 1$ case for a subgroup of G .

Assume $g(t_1, t_2)g^{-1} = (t'_1, t'_2) \in T^2$. Then $gt_2g^{-1} = t'_2$, so by the $n = 1$ case, there is some $g_0 \in N_G(T)$ with $g_0t_2g_0^{-1} = t'_2$. Let $g_1 = g_0^{-1}g$; then $g_1t_2g_1^{-1} = g_0^{-1}gt_2g^{-1}g_0 = g_0^{-1}t'_2g_0 = t_2$, so g_1 is in the centralizer $Z_G(t_2)$. The centralizer is a reductive group with maximal torus T . Let $t_1'' = g_0^{-1}t'_1g_0 = g_1t_1g_1^{-1}$. As the centralizer is a reductive group (although not necessarily connected), we can apply the $n = 1$ case again to get some element $g_2 \in N_{Z_G(t_2)}(T)$ with $g_2t_1g_2^{-1} = t_1''$. But through some rearrangement of the definition,

$$\begin{aligned} N_{Z_G(t_2)}(T) &= \{n \in Z_G(t_2) | nTn^{-1} = T\} = \{n \in G | nt_2n^{-1} = t_2, nTn^{-1} = T\} \\ &= N_G(T) \cap Z_G(t_2) \end{aligned}$$

Let $g' = g_0g_2$; an easy calculation shows that $g'(t_1, t_2)g'^{-1} = (t'_1, t'_2)$, and as g_0, g_2 are both in $N_G(T)$, the lemma is proven. \square

Define $X' = \{g | \forall i \, gt_i g^{-1} = t_i\} \subset X$; then $N_G(T) \cap Z_G(t_1, t_2, \dots, t_n)$ acts on X' . There is an obvious map $X'/(N_G(T) \cap Z_G(t_1, t_2, \dots, t_n)) \rightarrow X/N_G(T)$. Lemma 6 allows us to construct an inverse map, as it implies that any element of $X/N_G(T)$ has some representative in X' , so the two are isomorphic.

Therefore, we can rewrite $f^{-1}(t_1, t_2, \dots, t_n) = \{(g, t_1, t_2, \dots, t_n) | \forall i \, gt_i g^{-1} = t_i\}/(N_G(T) \cap Z_G(t_1, t_2, \dots, t_n))$. As the n -tuple in the numerator is now constant, this is isomorphic to $Z_G(t_1, t_2, \dots, t_n)/N_{Z_G(t_1, t_2, \dots, t_n)}(T)$.

We now have that for each $x \in X$, the fiber is isomorphic to the quotient of a reductive group by the normalizer of its maximal torus. By the same trick as in the beginning of lemma 6, this is isomorphic to the quotient of a connected reductive group (the connected component of the identity of the original group) by the normalizer of its maximal torus. This is exactly the situation referred to in Theorem 2 - so the fiber has trivial rational cohomology. This proves the theorem.

References

- [1] Brion, M.: Equivariant cohomology and equivariant intersection theory. arXiv:math/9802063